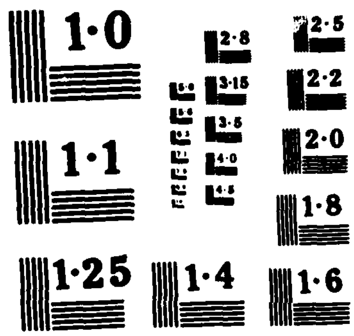
AD-A158 149 THE UNIQUENESS OF HILL'S SPHERICAL VORTEX(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER C J AMICK ET AL. MAY 85 MRC-TSR-2820 DAAG29-80-C-0041 1/1

UNCLASSIFIED F/G 12/1 NL

END
FILMED
DTIC


1·0
2·8
2·5
3·15
2·2
3·5
1·1
4·0
2·0
4·5
1·8
1·25
1·4
1·6

2

AD-A158 149

MRC Technical Summary Report #2820

THE UNIQUENESS OF HILL'S SPHERICAL VORTEX

C. J. Amick and L. E. Fraenkel

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

May 1985

(Received May 6, 1985)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

THE UNIQUENESS OF HILL'S SPHERICAL VORTEX

C.J. Amick[1] and L.E. Fraenkel[2]

Technical Summary Report # 2820
May 1985

ABSTRACT

The only explicit exact solution of the problem of steady vortex rings is that found, for a particular case, by M.J.M. Hill in 1894; it solves a semi-linear elliptic equation, of order two, involving a Stokes stream function psi $\psi(r,z)$ and a non-linearity sub H psi $f_H(\psi)$ that has a simple discontinuity at psi $\psi = 0$. In this paper we prove that (a) any weak solution of the corresponding boundary-value problem is Hill's solution, modulo translation along the axis of symmetry $(r = 0)$, (b) any solution of the isoperimetric variational problem in another paper ~~[12]~~ is a weak solution, indeed, any local maximizer is a weak solution. The result (b) is not immediate because f_H is discontinuous; consequently, the functional that is maximized is not Fréchet differentiable on the whole Hilbert space in question.

AMS (MOS) Subject Classifications: 76C05, 35R05, 35R35.

Key Words: steady vortex rings, weak solutions, spherical symmetry.

Work Unit Number 1 (Applied Analysis)

---

[1]Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637.

[2]Mathematics Division, University of Sussex, Brighton BN1 9QH, England.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

A number of existence theorems for steady vortex rings, and some properties of solutions, have been established in the last 15 years, but questions of uniqueness, and of any connection between the solutions resulting from different formulations, have remained very much open. It has not even been known whether the simple, explicit and celebrated solution known as Hill's spherical vortex is among those whose existence has been established by modern methods. The present paper settles this question; since Hill's vortex is shown to be unique, any existence theory that allows the discontinuous vorticity function in question recovers Hill's solution.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE UNIQUENESS OF HILL'S SPHERICAL VORTEX

C.J. Amick[1] and L.E. Fraenkel[2]

1. Introduction

1.1. Background

The mathematical description of steady vortex rings, in an ideal fluid occupying the whole space $\mathbb{R}^3$, can be approached in various ways. The physical basis of the problem, its history up to 1973, and several formulations are outlined in [12], pp. 14-21. Another, quite different formulation and the plan for a corresponding existence theory are presented in [6]. Further existence theorems, variational principles and results are to be found in [3], [7], [11], [13] and [20]. Here we state only definitions and equations that seem relevant to our immediate purpose.

Consider a Stokes stream function Ψ, defined on the closure $\bar{\Pi}$ of the half-plane

$$\Pi = \{(r,z) \mid r > 0, \ -\infty < z < \infty\},$$

where r and z may be regarded as cylindrical co-ordinates, points of $\mathbb{R}^3$ being denoted by $X = (X_1, X_2, X_3) = (r\cos\theta, r\sin\theta, z)$. The significance of Ψ is that (a) the fluid velocity $\underset{\sim}{q}$ has cylindrical components (in the directions r, θ, z increasing, respectively) $-\Psi_z/r, \ 0, \ \Psi_r/r$; (b) streamlines in a meridional plane (θ = const.) are level curves of Ψ, and $2\pi(\Psi_2 - \Psi_1)$

[1]Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637.

[2]Mathematics Division, University of Sussex, Brighton BN1 9QH, England.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

is the volumetric flow rate, or flux, between two stream surfaces of revolution described by $\Psi(r,z) = \text{const.} = \Psi_j$ $(j = 1,2)$; (c) the vorticity curl $\underset{\sim}{q}$ has cylindrical components $0, -(L\Psi)/r, 0$, where

$$L = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} .$$

Such a function Ψ corresponds to a steady vortex ring if there exists a bounded open set $A \subset \Pi$, called the cross-section of the ring and unknown a priori, such that $\Psi \in C^1(\bar{\Pi}) \cap C^2(\Pi\backslash\partial A)$ and satisfies the equations

$$L\Psi \equiv r\left(\frac{1}{r}\Psi_r\right)_r + \Psi_{zz} = \begin{cases} -\lambda r^2 f_0(\Psi) & \text{in } A , \\ 0 \text{ in } \Pi\backslash\bar{A} , \end{cases} \tag{1.1a}$$

$$\Psi|_{\partial A} = 0 , \qquad \Psi|_{r=0} = -k , \tag{1.1b,c}$$

$$\text{as } r^2 + z^2 \to \infty \text{ in } \bar{\Pi} , \ \Psi(r,z) \sim -\tfrac{1}{2}Wr^2 - k , \ \Psi_z/r \to 0$$

$$\text{and } \Psi_r/r \to -W . \tag{1.1d}$$

Here f_0 is a given, (strictly) positive vorticity function, which need be defined only on $(0,\infty)$ because of (1.2) below. We suppose for the moment that the vortex-strength parameter λ, the flux constant k and the propagation speed W are also prescribed, with $\lambda > 0$, $k \geq 0$ and $W > 0$. (The constant W is the speed of the vortex ring relative to the fluid at infinity; in (1.1) we have taken co-ordinate axes fixed in the ring and have demanded that the fluid velocity $\underset{\sim}{q} \to (0,0,-W)$ at infinity.) In most existence theorems, other sets of constants are prescribed, and 'free' elements of the set $\{\lambda,k,W\}$ are calculated a posteriori. This is illustrated by the remarks following (1.8) below.

Since $L\Psi < 0$ in A and $\Psi|_{\partial A} = 0$, the maximum principle implies that $\Psi > 0$ in A; similarly, $\Psi < 0$ in $\Pi\backslash\bar{A}$. Therefore we define the cross-section by

$$A = \{(r,z)\in\Pi \mid \Psi(r,z) > 0\} . \tag{1.2}$$

It is often convenient to write

$$\Psi(r,z) = \psi(r,z) - \tfrac{1}{2}Wr^2 - k , \tag{1.3}$$

where ψ is the stream function due to vorticity, while $-\tfrac{1}{2}Wr^2 - k$ represents a uniform stream. Note that the latter has zero vorticity: $L(\tfrac{1}{2}Wr^2 + k) = 0$. We define

$$f(t) = \begin{cases} 0 , & t \le 0 , \\ f_0(t) , & t > 0 . \end{cases}$$

Abbreviating the conditions at infinity, we now re-write (1.1) as

$$L\psi = -\lambda r^2 f(\Psi) \quad \text{in} \quad \Pi , \tag{1.4a}$$

$$\psi|_{r=0} = 0 , \quad \psi(r,z) \to 0 \quad \text{as} \quad r^2 + z^2 \to \infty \quad \text{in} \quad \bar{\Pi} , \tag{1.4b,c}$$

where it is to be understood that (1.4a) need not hold pointwise on ∂A, and that ψ_z/r and $\psi_r/r \to 0$ at infinity. Maximum principles for weak solutions show that $\psi > 0$ in Π.

1.2. Hill's spherical vortex

Only one explicit exact solution of (1.1) or (1.4) is known: that discovered by M.J.M. Hill [17] in 1894 for the case

$$k = 0 \quad \text{and} \quad f(t) = f_H(t) \equiv \begin{cases} 0 , & t \le 0 , \\ 1 , & t > 0 . \end{cases} \tag{1.5}$$

Hill observed that for this case a sphere $\{X \in \mathbb{R}^3 \mid |X| = a\}$ can serve as the boundary of a steady vortex 'ring' (Figure 1). Thus the cross-section is

$$A_H = \{(r,z) \in \Pi \mid r^2 + z^2 < a^2\} , \tag{1.6a}$$

and we let $\rho = (r^2 + z^2)^{\frac{1}{2}}$. Hill found the solution

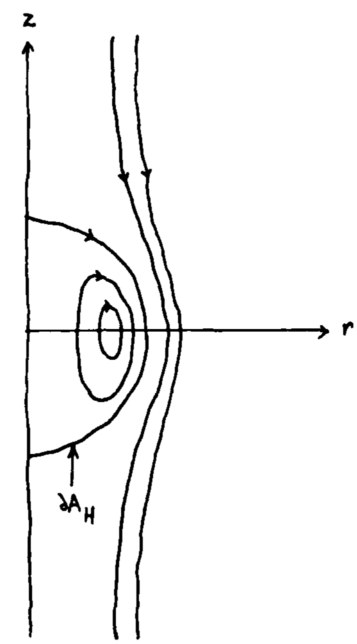


Figure 1. Hill's spherical vortex. The streamlines in Π are level curves of Ψ_H.

$$\psi_H(r,z) \equiv \Psi_H(r,z) + \tfrac{1}{2}Wr^2 = \begin{cases} \tfrac{1}{2}Wr^2\left(\frac{5}{2} - \frac{3}{2}\frac{\rho^2}{a^2}\right), & \rho \le a, \\ \tfrac{1}{2}Wr^2\,\frac{a^3}{\rho^3}, & \rho \ge a, \end{cases} \tag{1.6b}$$

where

$$\lambda a^2/W = 15/2 . \tag{1.6c}$$

Anticipating the definition (1.7), we note that, for a fluid of unit density, the kinetic energy $\pi\|\psi_H\|^2$ is given by

$$\|\psi_H\|^2 \equiv \int_\Pi \frac{1}{r^2}(\psi_{H,r}^2 + \psi_{H,z}^2)\, rdrdz = \frac{10}{7} W^2 a^3 . \tag{1.6d}$$

By <u>Hill's problem</u> we mean (1.1) or (1.4) for the case $k = 0$, $f = f_H$. Solutions of Hill's problem are presented (among other solutions) in [12] and [13], but no uniqueness theorem appears in these papers, and it has been unknown whether these solutions of Hill's problem are in fact Hill's solution. Indeed, we know of no result in the literature that connects in any way the many solutions, of the basic general problem (1.1), that have been obtained by different formulations and different existence theorems.

1.3. <u>Results</u>

In the present paper we prove that, for Hill's problem, (a) any weak solution is Hill's solution ψ_H, modulo translation in the z-direction; (b) any solution of the isoperimetric variational problem formulated in [12] is a weak solution, indeed, any <u>local</u> maximizer is a weak solution (and is therefore Hill's solution). The result (b) is not obvious or immediate, because f_H has a simple discontinuity; consequently, the functional that we maximize is not Fréchet differentiable on the whole Hilbert space appropriate to the problem.

We now make these statements precise. The Hilbert space $H(\Pi)$ is the completion of the set $C_0^\infty(\Pi)$, of real-valued functions having derivatives of every order and compact support in Π, in the norm $\|\cdot\|$ corresponding to the

inner product

$$\langle\phi,\chi\rangle = \int_{\Pi} \frac{1}{r^2} (\phi_r\chi_r + \phi_z\chi_z)\, rdrdz . \tag{1.7}$$

Thus $\pi\|\phi\|^2$ is the kinetic energy of the motion with stream function ϕ ; also,

$$\langle\phi,\chi\rangle = -\int_{\Pi} \frac{1}{r^2} \phi\, L\chi\, rdrdz \quad \text{if} \quad \phi,\chi \in C_0^\infty(\Pi) .$$

We shall say that ψ is a <u>weak solution of Hill's problem</u> if $\psi \in H(\Pi)\setminus\{0\}$ and if there exist constants $\lambda \in \mathbb{R}$ and $W > 0$ such that

$$\left.\begin{aligned} &\langle\phi,\psi\rangle = \lambda \int_{A(\psi)} \phi\, rdrdz \qquad \text{for all} \quad \phi \in H(\Pi) , \\ \text{where} \qquad &A(\psi) = \{(r,z)\in\Pi \mid \psi(r,z) > \tfrac{1}{2}Wr^2\} . \end{aligned}\right\} \tag{1.8}$$

Setting $\phi = \psi$ in (1.8), one sees that $\lambda > 0$ and that $A(\psi)$ must have positive area. (For Theorem 1.2, we shall prescribe $\|\psi\| > 0$ and $W > 0$; then $\lambda = \|\psi\|^2 \Big/ \int_{A(\psi)} \psi\, rdrdz$.) Our first result is

THEOREM 1.1. *If ψ is a weak solution of Hill's problem, then $\psi(r,z) = \psi_H(r,z-c)$ for some $c \in \mathbb{R}$; here ψ_H is as in* (1.6).

We now turn to the variational principle in [12]. To state it for Hill's problem, we define

$$t_+ = \int_0^t f_H(s)\,ds = \max\{t,0\} ,$$

$$J(\phi) = \int_{\Pi} (\phi(r,z) - \tfrac{1}{2}Wr^2)_+\, rdrdz \qquad \text{for all} \quad \phi \in H(\Pi) ,$$

$$S(\eta) = \{\phi \in H(\Pi) \mid \|\phi\|^2 = \eta > 0\} ;$$

the sphere $S(\eta)$ is a surface of constant energy. The variational problem is:

given W and η, find $\psi \in S(\eta)$ such that $J(\psi) = \max_{\phi \in S(\eta)} J(\phi)$. For the sake of a wide uniqueness statement, we consider not only solutions of this problem, but also local maximizers of J on $S(\eta)$; we shall prove

THEOREM 1.2. Let ψ be a local maximizer of J on $S(\eta)$: that is, $J(\phi) \le J(\psi)$ for all $\phi \in S(\eta)$ in some neighbourhood of ψ. Then ψ is a weak solution of Hill's problem (so that Theorem 1.1 applies).

1.4. Method

The principal steps in the paper are as follows.

(i) We make the transformation $\psi = r^2 v$ in order to prove that, when ψ is a weak solution of Hill's problem, v depends only on $\rho = \{r^2 + (z-c)^2\}^{1/2}$ for some $c \in \mathbb{R}$ (and on the parameters). The resulting one-dimensional problem for v can then be analysed without difficulty. It is a fortunate and crucial fact that

$$\psi = r^2 v \implies \frac{1}{r^2} L\psi = \frac{1}{r^3}(r^3 v_r)_r + v_{zz} = \Delta_5 v \quad \text{for } r > 0, \tag{1.9}$$

where $v(r,z)$ is now regarded as cylindrically symmetric in $\mathbb{R}^5$, that is, $r = (x_1^2 + \ldots + x_4^2)^{1/2}$ and $z = x_5$, while $\Delta_5 = \sum_{j=1}^5 \partial^2/\partial x_j^2$ is the Laplace operator in $\mathbb{R}^5$. The identity (1.9) was noted by Chandrasekhar ([9], p.252), and has been used by Ni [20] to prove regularity in his theory of steady vortex rings. However, Ni considered only non-linearities f smoother than f_H, so that Hill's vortex is outside the range of his theory. The transformation $\psi = r^2 v$ is exceptionally useful for Hill's problem because the equivalence, for $r > 0$, of the conditions $\psi > \frac{1}{2}Wr^2$ and $v > \frac{1}{2}W$ means that (1.4) becomes

$$\Delta_5 v = -\lambda f_H(v - \tfrac{1}{2}W) \quad \text{in } \mathbb{R}^5 \setminus \{r = 0\}, \tag{1.10a}$$

$$v(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{1.10b}$$

In Cartesian co-ordinates $x_1, \ldots, x_5$, equation (1.10a) has constant coefficients on both sides, and this allows us to prove, in step (ii), that v depends only on ρ. For $k \neq 0$ or $f \neq f_H$, the right-hand member of (1.4a) does not transform to a function independent of r.

(ii) To prove that the weak form of (1.10) implies spherical symmetry of the function v, we first transform (in section 2) the definition (1.8) of weak solution, showing at the same time that the exceptional line $\{r = 0\}$ causes no difficulty. In section 3, we establish regularity properties and positivity of the weak solution v, and then adapt to the present problem the powerful method initiated by Serrin in [24] and greatly enlarged by Gidas, Ni and Nirenberg in [14]. A slight, further extension is needed here because C^2 solutions are considered in [24] and [14], whereas for Hill's vortex second derivatives have a finite jump across ∂A_H. The generalized maximum principles in Gilbarg and Trudinger's book [15] enable us to modify the relevant parts of [14].

Different applications of the method in [24] and [14], to free-boundary problems of the same general kind as Hill's problem, have already been made in [8], [11] and [18].

(iii) To prove Theorem 1.2, we transform (in section 2) the variational principle for ψ to one for v. In section 4, we overcome the difficulty, that the convex functional N corresponding to J is not Fréchet differentiable on its whole domain, by using convex analysis. Detailed examination of the subdifferential of N, and of its left-hand and right-hand Gateaux derivatives, leads to the result.

1.5. Miscellaneous remarks

(i) The existence of a global maximizer, of J over the sphere $S(\eta)$, is not in doubt: such a function is constructed in [12], pp. 40-42 (although rather indirectly, by limiting procedures) and that particular function is

easily seen to be a weak solution, for any $k \geq 0$ and for a class of non-linearities f that includes f_H. For Hill's problem, one can also prove more directly, by means of the transformation $\psi = r^2 v$ and symmetrisation with respect to a point in $\mathbb{R}^5$ (that is, by re-arrangement of v to a spherically symmetric function v^*) that a global maximizer exists and is a weak solution. However, maximizers other than these are conceivable; it is for this reason, and to demonstrate the strength of the variational principle, that we present Theorem 1.2.

(ii) There is a variant of Hill's solution (not published, we believe, but well known to specialists) in which the fluid domain is a ball, say $\{x \in \mathbb{R}^3 \mid |x| < b\}$, with cross-section $D = \{(r,z) \in \Pi \mid r^2 + z^2 < b^2\}$. We set $k = 0$, $f = f_H$ as before; replace Π by D in (1.1a) and (1.4a); and replace the condition (1.1d) or (1.4c) at infinity by

$$\Psi(r,z) = -\tfrac{1}{2}Wr^2 \quad \text{on} \quad \partial D, \tag{1.11}$$

which states that the normal velocity on ∂D is the normal component of $(0,0,-W)$, and also implies the conditions (1.1c) and (1.4b) on $r = 0$ for this problem.

After extending in Appendix A the relevant theorem in [14], we show in Appendix B that the earlier results and methods carry over to this case with only minor changes. One of these is that, while transformed solutions v are always spherically symmetric, existence and uniqueness depend on what constants are given. If λ, W and b are prescribed, there may be no solution or there may be two; if $\|\psi\|$, W and b are prescribed, the solution always exists and is unique.

(iii) In [22], Norbury constructed perturbations of Hill's solution that represent genuine rings (homeomorphic to a solid torus). He solved (1.1) with $f = f_H$ and $0 < k < k_0$, where k_0 is small, by reducing an integral equation to a contraction mapping of a small closed ball, in a Banach space of functions

LEMMA 3.7. <u>Assume that for some</u> $\mu > 0$ <u>we have</u> $v(x) \geq v(x^\mu)$ <u>for all</u> $x \in Y(\mu)$, <u>and</u> $v(x_0) \neq v(x_0^\mu)$ <u>for some</u> $x_0 \in Y(\mu)$. <u>Then</u>

(a) $v(x) > v(x^\mu)$ <u>for all</u> $x \in Y(\mu)$, (3.12)

(b) $\frac{\partial v}{\partial x_1}(x) < 0$ <u>for all</u> $x \in T_\mu$. (3.13)

<u>Proof.</u> (a) We define the reflection in T_μ of any function F by $F_\mu(x) = F(x^\mu)$, and set $w = v_\mu - v$. By hypothesis, $w(x) \leq 0$ for $x \in Y(\mu)$, and we prove strict inequality by means of (3.1) and the maximum principle.

Let $Y = Y(\mu)$ and $Z = Z(\mu)$. Given $\phi \in C_0^\infty(Y)$, we note that ϕ_μ has support in Z, and choose $u = \phi_\mu$ in (3.1) to obtain

$$\int_Z \nabla\phi_\mu(z)\cdot\nabla v(z)\;dz \;=\; \lambda\int_{Z\cap P(v)} \phi_\mu(z)\;dz\,.$$

Set $z = x^\mu$ in this equation; then $x \in Y$, $\partial/\partial z_1 = -\partial/\partial x_1$ and $\partial/\partial z_j = \partial/\partial x_j$ for $j = 2,\ldots,5$. Also, $\phi_\mu(z) = \phi(z^\mu) = \phi(x)$ and $v(z) = v(x^\mu) = v_\mu(x)$. Consequently,

$$\int_Y \nabla\phi(x)\cdot\nabla v_\mu(x)\;dx \;=\; \lambda\int_{Y\cap P(v_\mu)} \phi(x)\;dx\,. \tag{3.14}$$

Now choose $u = \phi$ in (3.1), and subtract the resulting equation from (3.14) to obtain

$$\int_Y \nabla\phi\cdot\nabla w \;=\; \lambda\int_{Y\cap P(v_\mu)} \phi \;-\; \lambda\int_{Y\cap P(v)} \phi\,. \tag{3.15}$$

Since $v(x) > \frac{1}{4}W$ for $x \in P(v)$, while $v(x^\mu) > \frac{1}{4}W$ for $x \in P(v_\mu)$, our hypothesis implies that $Y \cap P(v)$ contains $Y \cap P(v_\mu)$; hence

$$\int_Y \nabla\phi\cdot\nabla w \leq 0 \qquad \text{for all} \quad \phi \in C_0^\infty(Y) \quad \text{with} \quad \phi \geq 0\,.$$

As it happens, we can now apply Theorem 3.4(a) to all of Y, because (3.10) shows that $v \in L_2(\mathbb{R}^5)$; then $w \in C(\bar{Y}) \cap W_2^1(Y)$, by Lemma 3.1, and so $\Delta w \geq 0$

Prospectus. It is easy to see that the function v_0, defined by (3.9), satisfies (3.1); our aim is to prove that $v_0(x) = v_H(|x'|, x_5)$. Having emphasized this, we shall omit the subscript 0 from v_0, restoring it only in the statement of our final result.

Lemmas 3.5, 3.6 and 3.8 below are merely statements for our case of Lemmas 4.1, 4.2 and 4.4 in [14]. We include these statements for the sake of clarity, but, apart from offering in Appendix C an alternative proof of Lemma 4.1 in [14], we refer to [14] for proofs of these results. The first part of the proof of Theorem 3.9 is also to be found in [14], but we include it as an essential part of the present story.

Reflecting hyperplanes. Let γ be a fixed unit vector in $\mathbb{R}^5$. For each $\mu \in \mathbb{R}$, define $T_\mu(\gamma) = \{x \in \mathbb{R}^5 \mid x\cdot\gamma = \mu\}$. We may suppose that, after a suitable rotation of axes, $\gamma = (1,0,\ldots,0)$; then

$$T_\mu = \{x \mid x_1 = \mu\} ; \tag{3.11a}$$

also,

$$x^\mu = (2\mu - x_1, x'') , \quad \text{where } x'' = (x_2,\ldots,x_5) , \tag{3.11b}$$

denotes the reflection in T_μ of any point x. We define open half-spaces by $Y(\mu) = \{x \mid x_1 < \mu\}$ and $Z(\mu) = \{x \mid x_1 > \mu\}$.

LEMMA 3.5. *Let v be as in (3.10), and consider two points y and z such that $y_1 < z_1$, $y_1 + z_1 \ge 2\mu > 0$ and $y'' = z''$. There exists a number $R(\mu)$, depending only on v and $\min\{1,\mu\}$, such that*

$$v(y) > v(z) \quad \text{whenever} \quad |y| \ge R(\mu) .$$

LEMMA 3.6. *There exists a number $\mu_0 \ge 1$ such that*

$$v(x) > v(x^\mu) \quad \text{whenever} \quad x \in Y(\mu) \quad \text{and} \quad \mu \ge \mu_0 .$$

where the symmetric $n \times n$ matrix $a(x)$ is uniformly positive definite:

$$\xi \cdot a(x) \cdot \xi \geq c_0 |\xi|^2, \quad c_0 = \text{const.} > 0, \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } x \in \Omega ;$$

the entries a_{ij} of $\underline{a}$ are in $W^1_\infty(\Omega)$, the components $p_1, \ldots, p_n$ of p are in $L_\infty(\Omega)$, and $q \in L_\infty(\Omega)$. Given a connected open (possibly unbounded) set $G \subset \Omega$, we say that $Lu \geq 0$ <u>weakly in</u> G if $u \in C(\bar{G}) \cap W^1_2(G)$ and

$$A(\phi, u; G) \equiv \int_G \{- \nabla\phi \cdot a(x) \cdot \nabla u + \phi p(x) \cdot \nabla u + \phi q(x) u\} \, dx \geq 0$$

for all $\phi \in C_0^\infty(G)$ with $\phi \geq 0$.

THEOREM 3.4. (a) <u>Let</u> $G \subset \Omega$ <u>be connected and open. If</u> $Lu \geq 0$ <u>weakly in</u> G <u>and</u> $u \leq 0$ <u>in</u> G, <u>then either</u> $u \equiv 0$ <u>or</u> $u < 0$ <u>in</u> G.

(b) <u>Let</u> $B \subset \Omega$ <u>be a ball, let</u> $x_0 \in \partial B$ <u>and let</u> m_0 <u>be a unit vector outward from</u> B <u>at</u> x_0. <u>(That is,</u> $m_0 \cdot (x_0 - c) > 0$, <u>where</u> c <u>is the centre of</u> B.) <u>If</u> $Lu \geq 0$ <u>weakly in</u> B, $u < 0$ <u>in</u> B, <u>and</u> $u(x_0) = 0$, <u>then</u>

$$\liminf_{t \to 0+} \frac{u(x_0) - u(x_0 - tm_0)}{t} > 0 ,$$

<u>which implies that</u> $m_0 \cdot (\nabla u)(x_0) > 0$ <u>when this derivative exists.</u>

<u>Proof.</u> (a) follows from Theorem 8.19 of [15]; (b) follows from the proof of Lemma 3.4 and (3.11) in [15] when the classical maximum principle used there is replaced by (a). The restriction $q \leq 0$ in Ω, imposed in [15], is not necessary for the particular conclusions in (a) and (b), because we can use a perturbation of our operator L that satisfies this restriction: if $A(\phi, u; G) \geq 0$ and $u \leq 0$ in G, then

$$\int_G \{- \nabla\phi \cdot a(x) \cdot \nabla u + \phi p(x) \cdot \nabla u + \phi q_-(x) u\} dx = A(\phi, u; G) - \int_G \phi q_+(x) u dx \geq 0$$

for all $\phi \in C_0^\infty(G)$ with $\phi \geq 0$. Q.e.d.

Remark. Following the procedure in [14], we now eliminate the dipole terms in (3.8) by defining (for all $x \in \mathbb{R}^5$)

$$v_0(x) = v(x+b), \quad \text{where} \quad b_j = a_j/3a_0 \quad (j = 1,\dots,5). \tag{3.9}$$

Then

$$\left.\begin{array}{l} v_0(x) = a_0|x|^{-3} + g(x), \qquad a_0 > 0, \\ |g(x)| \le \text{const.}|x|^{-5}, \quad |\nabla g(x)| \le \text{const.}|x|^{-6}, \end{array}\right\} \text{ for } |x| \ge 2R_v + |b|. \tag{3.10}$$

3.2. The maximum principle and reflecting hyperplanes

We note that v is a Newtonian potential with an unusual property: $v(x)$ is constant on the boundary $\partial P(v)$ of the set in which the density differs from zero. This is the underlying reason that $P(v)$ will turn out to be a ball, with v spherically symmetric about its centre (cf. [24]).

The method in [24] and [14] depends on moving hyperplanes in from infinity, reflecting the graph of a function about these hyperplanes, and then using the maximum principle. For positive solutions v of certain elliptic problems set in $\mathbb{R}^n$, the arguments in [14] are of two types: (a) those which depend only on approximations to $v(x)$ for large $|x|$, and (b) those which apply the maximum principle to classical solutions. Our result (3.8) is sufficient for (a), but we shall have to use (3.1) and a generalized maximum principle in place of (b).

The following maximum principle is far more general than is needed in this section, somewhat more general than is needed in Appendix A, and considerably less general than results in [15]. We state this particular theorem because it is close to the Maximum Principle and Lemma H on p.212 of [14]; thus it shows the feasibility of extending results in [14] to weak solutions of problems other than ours.

Let Ω be an open set in $\mathbb{R}^n$. Define, for $x \in \Omega$ and (say) $u \in C^2(\Omega)$,

$$Lu \equiv \nabla\cdot\{a(x)\cdot\nabla u\} + p(x)\cdot\nabla u + q(x)u,$$

$$- \int_{\mathbb{R}^5} \Delta K_n(y-x)\ v(y)\ dy = \lambda \int_{P(v)} K_n(x-y)\ dy\ . \tag{3.7}$$

Now $-\Delta K_n$ is a mollifying kernel: $\Delta K_n(z) = 0$ for $|z| \geq 1/n$, and

$$- \int_{B(0,1/n)} \Delta K_n(z)\,dz = 1\ , \qquad \int_{B(0,1/n)} |\Delta K_n(z)|\,dz \leq \text{const.},$$

where the constant is independent of n. Since $v \in C(\mathbb{R}^5)$, the left-hand member of (3.7) tends (pointwise) to $v(x)$ as $n \to \infty$. In addition,

$$\int_{\mathbb{R}^5} |K(z) - K_n(z)|\,dz < \frac{1}{6}\, n^{-2}\ ,$$

so that the right-hand member of (3.7) tends to the right-hand member of (3.6) as $n \to \infty$. Q.e.d.

LEMMA 3.3. <u>Let</u> R_v <u>be as in Lemma</u> 3.1(d). <u>There exist constants</u> $a_0 > 0$ <u>and</u> a_j $(j = 1,\ldots,5)$ <u>such that</u>

$$\left.\begin{aligned} v(x) &= a_0|x|^{-3} + \sum_{j=1}^{5} a_j x_j |x|^{-5} + h(x)\ , \\ |h(x)| &\leq \text{const.}|x|^{-5}\ , \quad |\nabla h(x)| \leq \text{const.}|x|^{-6}\ , \end{aligned}\right\} \quad \underline{\text{for}}\ \ |x| \geq 2R_v\ . \tag{3.8}$$

<u>In fact,</u>

$$a_0 = \frac{\lambda}{8\pi^2}\,|P(v)|_5\ , \qquad a_j = \frac{3\lambda}{8\pi^2}\int_{P(v)} y_j\ dy \qquad (j = 1,\ldots,5)\ ,$$

<u>but these details will not be needed.</u>

<u>Proof.</u> In (3.6), with $|y| < \frac{1}{2}|x|$ for all $y \in P(v)$, we may differentiate repeatedly under the integral sign (once is sufficient here) and expand $|y - x|^{-3}$ and its derivatives, essentially in powers of $|y|/|x|$, to finitely many terms with a remainder. Q.e.d.

$C^{1+\alpha}(\bar{B}_0)$, and this shows that $v(x) \to 0$ as $|x| \to \infty$.

(c) Since v is continuous in $\mathbb{R}^5$, the set $P(v)$ is open. Since $v \in W^2_{p,loc}(\mathbb{R}^5)$, we infer from (3.1) that

$$\int_{P(v)} \phi(\Delta v + \lambda) = 0 \qquad \text{for all} \quad \phi \in C_0^\infty(P(v)) \ ;$$

hence $-\Delta v = \lambda$ almost everywhere in $P(v)$. But the qualification 'almost everywhere' can be removed by means of further regularity theory, or by means of (3.6) below; in fact, v is real-analytic in $P(v)$. The argument is similar for the set in which $v(x) < \frac{1}{4}W$.

(d) This follows from (b) and the definition of $P(v)$. Q.e.d.

LEMMA 3.2. <u>The function</u> v <u>is the Newtonian potential of</u> $P(v)$ <u>with density</u> λ :

$$v(x) = \frac{\lambda}{8\pi^2} \int_{P(v)} |y - x|^{-3}\, dy \qquad \underline{\text{for all}} \quad x \in \mathbb{R}^5 . \tag{3.6}$$

<u>It follows that</u> $v > 0$ <u>in</u> $\mathbb{R}^5$.

<u>Proof.</u> We choose the test function u in (3.1) to be a smooth approximation to the Newtonian kernel in $\mathbb{R}^5$. Let μ be a non-decreasing function in $C^\infty(\mathbb{R} \to \mathbb{R})$ such that $\mu(t) = 0$ for $t \le \frac{1}{2}$ and $\mu(t) = 1$ for $t \ge 1$, and define, for all $z \in \mathbb{R}^5$ and any positive integer n,

$$K(z) = \frac{1}{8\pi^2} |z|^{-3} , \qquad K_n(z) = \mu(n|z|)K(z) .$$

We choose and fix any $x \in \mathbb{R}^5$, and replace the variable of integration in (3.1) by y. Then $K_n(\cdot - x) \in E$; we set $u(y) = K_n(y-x)$ in (3.1), and integrate the left-hand member by parts (first over a large ball $B(0,R)$; the boundary integral is at most $\max_{|y|=R} |v(y)|$ and tends to zero as $R \to \infty$, by Lemma 3.1(b)). Thus

$$M = \lambda|P(v) \cap B|_5^{1/p} \quad \text{and} \quad \frac{1}{p} + \frac{1}{p'} = 1 \; .$$

For any $x_0 \in \mathbb{R}^5$ and for $j = 0,\ldots,3$, let $B_j = \mathcal{B}(x_0,R_j)$ with $R_0 = \frac{1}{2}$, $R_1 = 1$, $R_2 = \frac{3}{2}$ and $R_3 = 2$. (These choices of R_j, and the inequalities (3.3) to (3.5), are dimensionally consistent in the physicist's sense only if we use non-dimensional variables, for example, if R_j is replaced by $R_j\lambda^{\frac{1}{2}}/W^{\frac{1}{2}}$ and v by v/W.) A first application of Agmon's theorem, with $p = 10/3$, yields

$$\| v \|_{2,10/3,B_1} \le k_3\{M_3(x_0) + \| v \|_{0,10/3,B_2}\} , \tag{3.3a}$$

where

$$M_3(x_0) = \lambda|P(v) \cap B_3|_5^{3/10} , \tag{3.3b}$$

and where k_3 is an absolute constant. Since $W^2_{10/3}(B_1)$ is embedded in the space $C(\bar{B}_1)$, a fortiori in $L_p(B_1)$ for all $p \ge 1$, a second application of Agmon's theorem gives

$$\| v \|_{2,p,B_0} \le k_2(p)M_2(x_0,p) + k_1(p)\{M_3(x_0) + \| v \|_{0,10/3,B_2}\} \quad \text{for all} \quad p \in (1,\infty) , \tag{3.4a}$$

where

$$M_2(x_0,p) = \lambda|P(v) \cap B_2|_5^{1/p} . \tag{3.4b}$$

Finally, embedding theory gives

$$\| v \|_{C^{1+\alpha}(\bar{B}_0)} \le k_0(p,\alpha)\| v \|_{2,p,B_0} \qquad \text{for} \quad p(1-\alpha) > 5 ; \tag{3.5}$$

the constants k_n depend only on p $(n = 1,2)$ or on p and α $(n = 0)$. To obtain bounds independent of x_0, we merely replace $|P(v) \cap B_j|_5$ by $|B_j|_5$ and the norm of v in $L_{10/3}(B_2)$ by that in $L_{10/3}(\mathbb{R}^5)$.

(b) Lemma 2.1 shows that the norm of v in $L_{10/3}(\mathbb{R}^5 \setminus \mathcal{B}(0,m))$, and $|P(v)\setminus\mathcal{B}(0,m)|_5$, both tend to zero as $m \to \infty$. Hence the right-hand member of (3.4a) tends to zero as $|x_0| \to \infty$; so, therefore, does the norm of v in

$|P(v)|_5 > 0$. <u>It is to be understood throughout section</u> 3 <u>that</u> v, λ <u>and</u> W <u>have these properties.</u>

<u>Notation.</u> We define <u>balls</u> to be <u>open</u> and of positive, finite radius; $\mathcal{B}(c,R)$ will denote the ball with centre c and radius R in the space implied by the context. The non-negative and non-positive parts of a real-valued function are defined, respectively, by

$$g_+(x) = \max\{g(x),0\}, \qquad g_-(x) = \min\{g(x),0\}; \tag{3.2}$$

note that, in contrast to the convention in integration theory, the non-positive part is non-positive. Since sections 3 and 4 concern only statements (I) and (II), we can safely ignore two previous conventions: the Δ_5 in (1.9) now becomes Δ, and (as is usual) ϕ will be used for smooth test functions rather than for elements of $H(\Pi)$.

LEMMA 3.1. (a) $v \in W^2_{p,loc}(\mathbb{R}^5) \cap C^{1+\alpha}(\mathbb{R}^5)$ <u>for all</u> $p \in (1,\infty)$ <u>and</u> $\alpha \in (0,1)$.

(b) $v(x) \to 0$ <u>(pointwise) as</u> $|x| \to \infty$.

(c) $$-\Delta v(x) = \begin{cases} \lambda & \underline{\text{in}} \quad P(v), \\ 0 & \underline{\text{in}} \quad \{x \in \mathbb{R}^5 \,|\, v(x) < \tfrac{1}{2}W\}. \end{cases}$$

(d) <u>There exists a number</u> R_v <u>such that</u> $P(v) \subset \mathcal{B}(0,R_v)$ <u>in</u> $\mathbb{R}^5$.

<u>Proof.</u> (a) We use Agmon's L_p approach to the Dirichlet problem, applying Theorem 6.1 of [1] to the operator Δ . The hypotheses of that theorem are amply satisfied, because $v \in L_{10/3}(\mathbb{R}^5)$ and because (3.1) implies that, for any ball B in $\mathbb{R}^5$ and any $p \in (1,\infty)$,

$$\left|\int_B v\Delta\phi\right| = \lambda\left|\int_{P(v)\cap B} \phi\right| \le M\|\phi\|_{0,p',B} \quad \text{for all} \quad \phi \in C_0^\infty(B),$$

where

the first integral vanishes because $v \in E_c$, and the second vanishes by (2.4) and Lemma 2.3. Q.e.d.

2.3. Transformation of the variational principle

Given $W > 0$, we define

$$N(u) = \frac{1}{2\pi^2}\int_{\mathbb{R}^5} (u - \tfrac{1}{4}W)_+ \, dx = \frac{1}{2\pi^2}\int_{P(u)} (u - \tfrac{1}{4}W)\, dx \quad \text{for all } u \in E, \tag{2.8}$$

$$S_c(\eta) = \{u \in E_c \mid |||u|||^2 = \eta > 0\} . \tag{2.9}$$

Under the isomorphism in Lemma 2.2, the variational problem stated before Theorem 1.2 becomes: given W and η, find $v \in S_c(\eta)$ such that $N(v) = \max_{u \in S_c(\eta)} N(u)$. Thus we have

THEOREM 2.5. Theorem 1.2 is equivalent to the following statement.

(II) Let v be a local maximizer of N on $S_c(\eta)$. Then v is a *transformed* weak solution of Hill's problem; that is, there exists $\lambda \in \mathbb{R}$ such that (2.7) holds.

3. Transformed weak solutions correspond to Hill's vortex

3.1. Preliminary estimates

Here and in section 3.2, we prove the truth of statement (I) in Theorem 2.4. In fact, we prove a little more: that, if the hypothesis $v \in E_c$ in (I) is weakened to $v \in E$, then the conclusion still holds, provided that $v_H(|x'|, x_5 - c)$, for some $c \in \mathbb{R}$, is replaced by $v_H(|x' - b'|, x_5 - b_5)$, for some $b \in \mathbb{R}^5$. Thus our hypothesis is: there exist $v \in E\backslash\{0\}$, $\lambda \in \mathbb{R}$ and $W > 0$ such that

$$\int_{\mathbb{R}^5} \nabla u \cdot \nabla v \, dx = \lambda \int_{P(v)} u \, dx \qquad \text{for all } u \in E, \tag{3.1}$$

where $P(v)$ is as in (2.2). Setting $u = v$, we see that $\lambda > 0$ and

$$\langle u_0, w_1\rangle_E = \frac{1}{2\pi^2}\int_\Pi r^3 drdz \int_{S^3} (u_{0,r}w_{1,r} + u_{0,z}w_{1,z})\, d\omega_\xi$$

$$= \int_\Pi \{u_{0,r}M(w_{1,r}) + u_{0,z}M(w_{1,z})\}\, r^3 drdz = 0 .$$

Since $|||Mu||| = |||u_0||| \le |||u|||$ for all $u \in C_0^\infty(\mathbb{R}^5)$, we extend M by continuity to have domain E; then the result $\langle Mu, w - Mw\rangle_E = 0$ extends to all $u,w \in E$, so that M and $I - M$ (where I denotes the identity operator on E) are respectively the projection operators of E onto E_c and onto $E_c^\perp$. Equation (2.5) now follows from the bound

$$\left|\int_F u\, dx\right| = \left|\int_F Mu\, dx\right| \le c_1 |||Mu|||\ |F|_5^{7/10} \qquad \text{for all} \quad u \in E ,$$

implied by (2.3), and from the fact that $Mw = 0$ whenever $w \in E_c^\perp$. Q.e.d.

We are now in a position to state our first objective in terms of functions in E.

THEOREM 2.4. Theorem 1.1 is equivalent to the following statement.

(I) If there exist $v \in E_c\backslash\{0\}$, $\lambda \in \mathbb{R}$ and $W > 0$ such that

$$\int_{\mathbb{R}^5} \nabla u\cdot\nabla v\, dx = \lambda \int_{P(v)} u\, dx \qquad \text{for all} \quad u \in E, \tag{2.7}$$

where $P(v)$ is as in (2.2), then $v(x) = v_H(|x'|, x_5 - c)$ for some $c \in \mathbb{R}$. Here $v_H(r,z) = \psi_H(r,z)/r^2$ and ψ_H is as in (1.6).

Proof. For any $u \in E$, we use the decomposition $u = u_0 + u_1$, where $u_0 \in E_c$ and $u_1 \in E_c^\perp$. Let (I_0) denote the variant of statement (I) that results from replacing $u \in E$ by $u_0 \in E_c$ in (2.7). Lemma 2.2 shows that Theorem 1.1 and (I_0) are equivalent. Moreover, (I_0) and (I) are equivalent because

$$\int_{\mathbb{R}^5} \nabla u_1\cdot\nabla v = 0 \quad \text{and} \quad \lambda\int_{P(v)} u_1 = 0 \qquad \text{for all} \quad u_1 \in E_c^\perp ;$$

For any $u \in C^{\infty}_{0,c}(\mathbb{R}^5)$ and any positive integer n, define $u_n \in C^{\infty}_d(\mathbb{R}^5)$ by $u_n(r,z) = \mu(nr)u(r,z)$. A calculation shows that $|||u - u_n||| \le \text{const.}\ n^{-1}$, where the constant depends on u but not on n; thus $C^{\infty}_d(\mathbb{R}^5)$ is dense in $C^{\infty}_{0,c}(\mathbb{R}^5)$ under the norm $|||\cdot|||$, and hence in E_c.

Now let $\phi = r^2u$ and $\chi = r^2w$, where ϕ and χ are in $C^{\infty}_0(\Pi)$ or, equivalently, u and w are in $C^{\infty}_d(\mathbb{R}^5)$. From (1.7) and (2.1) we find that

$$\langle\phi,\chi\rangle_{H(\Pi)} = \int_{\Pi} \{r^3(u_r w_r + u_z w_z) + 2(r^2uw)_r\}drdz = \langle u,w\rangle_E ,$$

since the integral of $2(r^2uw)_r$ vanishes. Q.e.d.

LEMMA 2.3. <u>Let</u> F <u>be a figure of revolution having finite measure</u>: $|F|_5 < \infty$. <u>Let</u> $E_c^{\perp}$ <u>denote the orthogonal complement in</u> E <u>of</u> E_c. <u>Then</u>

$$\int_F w\,dx = 0 \qquad \underline{\text{for all}} \quad w \in E_c^{\perp}. \tag{2.5}$$

<u>Proof.</u> Any $u \in C^{\infty}_0(\mathbb{R}^5)$ has values $u(r,\xi,z)$, where $\xi \in S^3$; we define a mean-value operator M by

$$(Mu)(r,z) = \frac{1}{2\pi^2}\int_{S^3} u(r,\xi,z)\,d\omega_{\xi} ,$$

where $d\omega_{\xi}$ denotes the element of surface area at ξ, and decompose u as follows:

$$u = u_0 + u_1 , \qquad \text{where } u_0 = Mu , \quad u_1 = u - Mu . \tag{2.6}$$

Then $u_0 \in E_c$, $Mu_1 = 0$ and $u_1(r,\xi,z) = O(r)$ as $r \to 0$. We now show that $\langle u_0,w_1\rangle_E = 0$ for all $u,w \in C^{\infty}_0(\mathbb{R}^5)$; hence (2.6) corresponds, for such functions, to the unique orthogonal decomposition $E = E_c \oplus E_c^{\perp}$. Differentiating the equation $Mw_1 = 0$, we obtain $M(w_{1,r}) = (Mw_1)_r = 0$ and $M(w_{1,z}) = 0$. Consequently,

$$\| u \|_{m,p,\Omega} = \left\{ \sum_{|\alpha| \le m} \int_\Omega |D^\alpha u|^p \right\}^{1/p} .$$

Note that $\| \cdot \|_{0,p,\Omega}$ denotes the norm of $L_p(\Omega)$.

2.2. Transformation of weak solutions

We begin with three simple lemmas that establish basic properties of functions in E .

LEMMA 2.1. (a) The space E is embedded in $L_{10/3}(\mathbb{R}^5)$, and

$$\| u \|_{0,10/3,\mathbb{R}^5} \le c_1 |||u||| , \quad \text{where } c_1 = \left(\tfrac{2}{5}\right)^{\frac{1}{2}} \tfrac{4}{3}\pi , \text{ for all } u \in E . \tag{2.3}$$

(b) With $P(u)$ as in (2.2),

$$|P(u)|_5 \le c_2 W^{-10/3} |||u|||^{10/3} , \text{ where } c_2 = (2c_1)^{10/3} , \text{ for all } u \in E . \tag{2.4}$$

Proof. The first inequality is a standard result of Sobolev embedding ([21], p.128), combined with the inequality between the geometric mean and the root mean square. The second inequality then follows from

$$\int_{P(u)} u^{10/3} \, dx \ge (\tfrac{1}{2}W)^{10/3} |P(u)|_5 . \qquad \text{Q.e.d.}$$

LEMMA 2.2. The spaces $H(\Pi)$ and E_c are isometrically isomorphic under the transformation $\phi = r^2 u$ of any $\phi \in H(\Pi)$ or $u \in E_c$.

Proof. Let $C_d^\infty(\mathbb{R}^5)$ denote the set of functions in $C_{0,c}^\infty(\mathbb{R}^5)$ that have support disjoint from the z-axis (here $r = |x'|$ and $z = x_5$) . First we show that $C_d^\infty(\mathbb{R}^5)$ is dense in E_c . Let μ be a non-decreasing function in $C^\infty(\mathbb{R} \to \mathbb{R})$ such that $\mu(t) = 0$ for $t \le \frac{1}{2}$ while $\mu(t) = 1$ for $t \ge 1$.

will be called a *figure of revolution*, or cylindrically symmetric set, with cross-section X. The subscript c, attached to the symbol for a set of functions, denotes the subset of cylindrically symmetric functions; that is, of functions u such that $u(x) = \hat{u}(|x'|,x_5)$ for all x in the cylindrically symmetric domain of u. We shall sometimes write u for $\tilde{u}$. The closed linear subspace of E, formed by completing $C^{\infty}_{0,c}(\mathbb{R}^5)$ in the norm $|||\cdot|||$, will be denoted by E_c.

By the transformation $\phi = r^2u$ we mean that, given $\phi \in H(\Pi)$, we define $u : \mathbb{R}^5 \setminus \{x' = 0\} \to \mathbb{R}$ by $u(x) = \phi(|x'|,x_5)/|x'|^2$, or that, given $u \in E_c$ with $u(x) = \tilde{u}(|x'|,x_5)$, we define $\phi : \Pi \to \mathbb{R}$ by $\phi(r,z) = r^2\tilde{u}(r,z)$. Since $\mathbb{R}^3$ does not occur in this statement, we now write $r = |x'|$, $z = x_5$ with no danger of confusion. Note that

$$\langle u,w\rangle_E = \int_{\Pi} (u_r w_r + u_z w_z) r^3 drdz \quad \text{if} \quad u,w \in E_c, \tag{2.1}$$

because the three-dimensional unit sphere $S^3 = \{y \in \mathbb{R}^4 \,|\, |y| = 1\}$ has area $2\pi^2$.

Given a constant $W > 0$, we define

$$P(u) = \{x \in \mathbb{R}^5 \,|\, u(x) > \tfrac{1}{2}W\} \quad \text{for any} \quad u \in E. \tag{2.2}$$

Of course, the elements of E are really equivalence classes (of functions equal almost everywhere), and the precise form of $P(u)$ depends on the representative u selected from an equivalence class $[u] \in E$, but different representatives only change $P(u)$ by sets of (five-dimensional Lebesgue) measure zero. If $u \in E_c$ and $\phi = r^2u$, then $P(u)$ is the figure of revolution with cross-section $A(\phi)$ [*cf.* (1.8)].

For sets, $|\cdot|_n$ denotes n-dimensional Lebesgue measure.

For any open set $\Omega \subset \mathbb{R}^n$, $W^m_p(\Omega)$ denotes the Sobolev space of functions having generalized derivatives up to order m in $L_p(\Omega)$, $p \ge 1$; its norm will be written

capable of representing the unknown boundary ∂A. We shall prove in [4] that Norbury's solutions are also unique for sufficiently small (positive) values of k. This is not trivial because, in Norbury's Banach space, the closed ball forming the domain of his contraction mapping necessarily has a radius that tends to zero as $k \to 0$. Thus, for small $k > 0$, there could exist solutions close to Hill's solution that are outside the range of the local uniqueness result in [22]. It is reassuring that, in fact, a single branch of solutions emerges from Hill's solution as the parameter k increases from zero. Norbury's numerical calculations [23] suggest that this branch is defined for all $k > 0$, and represents rings of small cross-section as $k \to \infty$.

(iv) A small third contribution, in our endeavour to unify the diverse theories of steady vortex rings, will be presented in [5]. There we consider (1.1) with $k = 0$ and the power-law vorticity function $f_0(t) = t^\beta$, $\beta = \text{const.} \in (0,5)$, and prove that for these cases the solutions in [12] coincide with those found by a wholly different variational principle in [13]. This is not a uniqueness result, but merely a proof of the equivalence of two different methods.

2. The transformed problem

2.1. Further notation and terminology

We define the Hilbert space E to be the completion of the set $C_0^\infty(\mathbb{R}^5)$ in the norm $|||\cdot|||$ corresponding to the inner product

$$\langle u,w\rangle_E = \frac{1}{2\pi^2}\int_{\mathbb{R}^5} \nabla u(x)\cdot\nabla w(x)\,dx = \frac{1}{2\pi^2}\int_{\mathbb{R}^5} \nabla u\cdot\nabla w\ ;$$

abbreviations as in the last expression will be used where no confusion can arise.

Any set of the form

$$F = \{x \in \mathbb{R}^5 \,|\, (|x'|,x_5) \in X \subset \bar{\Pi}\}, \quad \text{where} \quad x' = (x_1, \ldots, x_4),$$

weakly in Y. Since $w \not\equiv 0$ in Y, by hypothesis, we conclude that $w < 0$ in Y. (The appeal to (3.10) is not necessary. The set $X \equiv \{x \in Y | w(x) = 0\}$ is closed in Y because w is continuous, and open in Y by application of Theorem 3.4(a) to a small ball about any zero of w in Y. Therefore $X = \emptyset$ or $X = Y$, and the latter is contrary to hypothesis.)

(b) For any $x_0 \in T_\mu$ and any $R > 0$, define $B = \mathcal{B}((x_{0,1} - R, x_0''), R)$, so that $B \subset Y$ and $\partial B \cap T_\mu = \{x_0\}$. Then $w < 0$ in B, and $w(x_0) = 0$; by Theorem 3.4(b), $(\partial w/\partial x_1)(x_0) > 0$, since Lemma 3.1(a) ensures that this derivative exists. Finally, $(\partial w/\partial x_1)(x_0) = -2(\partial v/\partial x_1)(x_0)$. Q.e.d.

LEMMA 3.8. <u>The set</u> $\{\mu > 0 | v(x) > v(x^\mu)$ <u>for all</u> $x \in Y(\mu)\}$ <u>is open in</u> $\mathbb{R}$.

THEOREM 3.9. <u>Let</u> v, λ <u>and</u> W <u>be as in</u> (3.1), <u>and</u> v_0 <u>as in</u> (3.9). <u>Then</u> $v_0(x) = v_H(|x'|, x_5)$, <u>where</u> $v_H(r,z) = \psi_H(r,z)/r^2$ <u>and</u> ψ_H <u>is as in</u> (1.6).

<u>Proof.</u> Only v_0 is discussed in this proof; we continue to abbreviate v_0 to v. Let (m,∞), with $m \geq 0$, be the maximal open interval such that (3.12) and (3.13) hold whenever $\mu \in (m,\infty)$. That such an interval exists follows from Lemma 3.6 and the fact that (3.12) implies (3.13). If $m > 0$, then by continuity $v(x) \geq v(x^m)$ for all $x \in Y(m)$, and by Lemma 3.5 there exist points $x_0 \in Y(m)$ such that $v(x_0) > v(x_0^m)$. Hence Lemma 3.7 is applicable and shows that (3.12) and (3.13) hold for $\mu \geq m$; by Lemma 3.8, (m,∞) is not maximal. We conclude that $m = 0$, whence $v(-x_1, x'') \geq v(x_1, x'')$ whenever $x_1 \geq 0$. Repeating the argument for the unit vector $\gamma = (-1,0,\ldots,0)$, we see that v is an even function of x_1. Also, $(\partial v/\partial x_1)(x) < 0$ whenever $x_1 > 0$, because (3.13) holds for all $\mu > 0$. The same argument holds for every unit vector γ in $\mathbb{R}^5$; therefore, v depends only on $|x|$ and is strictly decreasing as $|x|$ increases.

It follows that $P(v) = \bar{B}(0,a)$ for some $a > 0$. Let $|x| = \rho$ and $v(x) = \tilde{v}(\rho)$; from Lemma 3.1 we infer that $\tilde{v} \in C^1[0,\infty)$, that $\tilde{v}$ is real-analytic in $[0,a)$ and (a,∞), and that

$$\frac{1}{\rho^4}\frac{d}{d\rho}\left(\rho^4 \frac{d}{d\rho}\right)\tilde{v} = \begin{cases} -\lambda & \text{for } 0 < \rho < a, \\ 0 & \text{for } \rho > a, \end{cases}$$

$$\tilde{v}(a) = \tfrac{1}{4}W, \qquad \tilde{v}(\rho) \to 0 \quad \text{as } \rho \to \infty .$$

This problem can be solved explicitly and easily; the solution corresponds to (1.6b) and (1.6c). (In fact, we can reach this conclusion with less a priori knowledge of $\tilde{v}$; once spherical symmetry is established, the maximum principle ensures that $P(v)$ must be a ball about the origin, otherwise $\tilde{v}$ would have a local minimum.) Q.e.d.

4. Local maximizers of N on $S_c(\eta)$

In this section, recalling that

$$N(u) = \frac{1}{2\pi^2}\int_{\mathbb{R}^5} (u - \tfrac{1}{4}W)_+ \, dx = \frac{1}{2\pi^2}\int_{P(u)} (u - \tfrac{1}{4}W)\, dx \quad \text{for all } u \in E,$$

we prove statement (II) in Theorem 2.5: that a local maximizer of N on the sphere $S_c(\eta)$ in E_c is a weak solution in the sense of (2.7). If we widen the question by considering N on the corresponding sphere, say $\Sigma(\eta)$, in E, then the same analysis shows that a local maximizer of N on $\Sigma(\eta)$ is a weak solution in the sense of (3.1). However, it is not obvious that a local maximizer on $S_c(\eta)$ (arising from Hill's problem set in Π) is a local maximizer on the bigger sphere $\Sigma(\eta)$, and it does not seem worth while to pursue this point.

The functional N is not Gateaux differentiable, let alone Fréchet differentiable, at all points of the space E (see the remark following Lemma 4.2 below). Hence it is not obvious that any local maximizer of N on $S_c(\eta)$

is a weak solution, and (as was mentioned in (iii) of section 1.4) we use convex analysis to prove this result. However, convex analysis is not needed for the global maximizers described in (i) of section 1.5; in particular, the method of spherical re-arrangement (which conserves $N(v)$ and does not increase $|||v|||$) leads directly to a weak solution of Hill's problem.

Before coming to the statement (II), we derive relevant properties of the functional N, and these are established without restricting N to E_c.

LEMMA 4.1. <u>The functional</u> N <u>is convex, bounded by</u>

$$0 \le N(u) \le \text{const. } W^{-7/3} \, |||u|||^{10/3} \qquad \underline{\text{for all}} \quad u \in E\,, \tag{4.1}$$

<u>and locally Lipschitz continuous:</u>

$$|N(u) - N(w)| \le \text{const. } W^{-7/3} R^{7/3} \, |||u-w||| \quad \underline{\text{for all}} \quad u,w \in B(0,R) \ \underline{\text{in}} \ E\,. \tag{4.2}$$

<u>Here the constants are independent of</u> u, w, W <u>and</u> R.

<u>Proof.</u> The convexity of N follows from that of the function $(\cdot)_+$; for $t \in [0,1]$,

$$\begin{aligned} N((1-t)u + tw) &= \frac{1}{2\pi^2}\int_{\mathbb{R}^5} \{(1-t)(u - \tfrac{1}{2}W) + t(w - \tfrac{1}{2}W)\}_+ \\ &\le \frac{1}{2\pi^2}\int_{\mathbb{R}^5} \{(1-t)(u - \tfrac{1}{2}W)_+ + t(w - \tfrac{1}{2}W)_+\} \\ &= (1-t)N(u) + tN(w)\,. \end{aligned}$$

The bound (4.1) results from Hölder's inequality and (2.3), (2.4). Lipschitz continuity of bounded convex functionals is a standard result ([10], pp. 12-13; [16], pp. 110, 113), but it seems worth while to give the short proof of (4.2).

Let $h = |||w-u||| > 0$ and $q = w + R(w-u)/h$. Then

$$w = \frac{h}{R+h} q + \frac{R}{R+h} u \quad \Longrightarrow \quad N(w) \le \frac{h}{R+h} N(q) + \frac{R}{R+h} N(u)$$

by convexity, and $q \in \bar{B}(0,2R)$. Accordingly,

$$N(w) - N(u) \le \frac{h}{R+h} N(q) - \frac{h}{R+h} N(u) \le \frac{h}{R} N(q) \le \text{const. } h\, W^{-7/3} R^{7/3} ,$$

by (4.1). We obtain a similar inequality by interchanging w and u, and (4.2) follows. Q.e.d.

At any point $v \in E$, the <u>right-hand Gateaux derivative</u> of N in the direction u is defined by

$$N'_+(v)u = \lim_{t \to 0+} \frac{N(v+tu) - N(v)}{t} , \qquad u \in E ;$$

the left-hand derivative $N'_-(v)u$ is defined similarly with $t \to 0-$. It is to be expected from Lemma 4.1, and is true ([16], p.117), that these limits *exist for* all v and u in E, that $N'_-(v)u = -N'_+(v)(-u)$ and that $N'_+(v)$ is a sublinear functional on E . We now calculate these one-sided derivatives.

LEMMA 4.2. <u>At any</u> $v \in E$ <u>we have, respectively,</u>

$$2\pi^2 N'_\pm(v)u = \int_{P(v)} u(x)\,dx + \int_{X(v)} u_\pm(x)\,dx \qquad \underline{\text{for all}} \quad u \in E ,$$

<u>where</u> $P(v)$ <u>is as in</u> (2.2), u_+ <u>and</u> u_- <u>are as in</u> (3.2), <u>and</u> $X(v) = \{x \mid v(x) = \tfrac{1}{4}W\}$.

<u>Proof.</u> It is sufficient to prove the result for $N'_+(v)$. We abbreviate $P(v)$ to P and $X(v)$ to X, let $V(x) = v(x) - \tfrac{1}{4}W$, and define

$$Q(t) = P(v+tu) = \{x \mid V(x) + tu(x) > 0\} , \qquad t > 0 .$$

Then

$$2\pi^2 \; \frac{N(v+tu) - N(v)}{t} = \frac{1}{t}\left\{\int_{Q(t)} (V + tu) - \int_P V\right\}$$

$$= \int_{Q(t)\cap P} u + \frac{1}{t}\int_{Q(t)\setminus P} (V+tu) - \frac{1}{t}\int_{P\setminus Q(t)} V \,. \qquad (4.3)$$

We consider these integrals one at a time, always taking fixed, but arbitrary, representatives v, u of the equivalence classes $[v],[u] \in E$.

(i) First,

$$\int_{Q(t)\cap P} u = \int_P u - \int_{D(t)} u \,, \qquad (4.4)$$

where

$$D(t) = P\setminus Q(t) = \{x \mid V(x) > 0 \,,\; V(x) + tu(x) \le 0\} \,,$$

so that $u < 0$ on $D(t)$. Consequently, $D(s) \subset D(t)$ for $0 < s \le t$, because $V(x) \le s|u(x)|$ implies that $V(x) \le t|u(x)|$. For any $x \in P$, define $t_0(x) = V(x)/|u(x)| > 0$; then $x \notin D(t)$ for $t < t_0(x)$. Hence $|D(t)|_5 \to 0$ as $t \to 0+$, and

$$\left|\int_{D(t)} u\right| \le \|u\|_{0,10/3,\mathbb{R}^5} \; |D(t)|_5^{7/10} \to 0 \quad \text{as} \quad t \to 0+ \,. \qquad (4.5)$$

(ii) Since

$$Q(t)\setminus P = \{x \mid V(x) + tu(x) > 0 \,,\; V(x) \le 0\} \,,$$

we have $u > 0$ on $Q(t)\setminus P$, and $Q(t)\setminus P = \{x|u(x) > 0 \,,\; V(x) = 0\} \cup R(t)$, where

$$R(t) = \{x \mid V(x) + tu(x) > 0 \,,\; V(x) < 0\} \,.$$

Then $|R(t)|_5 \to 0$ as $t \to 0+$, by the reasoning used for $D(t)$ in (i), and

$$\frac{1}{t}\int_{Q(t)\setminus P} (V+tu) = \int_X u_+ + \frac{1}{t}\int_{R(t)} (V+tu) \,, \qquad (4.6)$$

where

$$0 \le \frac{1}{t}\int_{R(t)} (v + tu) \le \int_{R(t)} u \to 0 \quad \text{as} \quad t \to 0+ . \tag{4.7}$$

(iii) The third integral in (4.3) is over $D(t)$, already considered in (i), and

$$0 \le \frac{1}{t}\int_{P\setminus Q(t)} v \le \int_{D(t)} |u| \to 0 \quad \text{as} \quad t \to 0+ . \tag{4.8}$$

Combining (4.4) to (4.8), we obtain the result of the lemma. Q.e.d.

Lemma 4.2 shows that N is Gateaux differentiable at v in all directions (that is, $N'_+(v) = N'_-(v)$) if and only if the set $\{x \mid v(x) = \tfrac{1}{2}W\}$ has measure zero.

The subdifferential of N at any point $v \in E$ is the set

$$\partial N(v) = \{g^* \in E^* \mid g^*(u-v) \le N(u) - N(v) \text{ for all } u \in E\}$$

in the dual space E^* of E. The following is another standard result ([16], p.122).

LEMMA 4.3. *At any point* $v \in E$, *a bounded linear functional* $g^* \in \partial N(v)$ *if and only if*

$$N'_-(v)u \le g^*(u) \le N'_+(v)u \quad \textit{for all} \quad u \in E .$$

We now consider the restriction $N|_{E_c} = N_c$, say. (The subscript c is redundant in some statements, but helpful in others.) It is clear that N_c enjoys the same properties relative to E_c as N does relative to E. The following lemma provides a generalization of the usual Euler-Lagrange equation, in weak form, characterizing a local maximum.

LEMMA 4.4. *Let* v *be a local maximizer of* N_c *on the sphere* $S_c(\eta)$ *(that is,* $v \in S_c(\eta)$ *and* $N_c(u) \le N_c(v)$ *for all* $u \in S_c(\eta)$ *in some neighbourhood of* v) .

If $g^* \in \partial N_c(v)$, *then* $g^* = \alpha\langle v,\cdot\rangle_E$ *for some* $\alpha > 0$.

Proof. There exists a unique element $g \in E_c$ such that $g^* = \langle g,\cdot\rangle_E$, and g has a unique decomposition $g = \alpha v + w$, where $\alpha \in \mathbb{R}$, $w \in (\text{span}\,\{v\})^\perp$ and $(\cdot)^\perp$ denotes the orthogonal complement in E_c . We prove that $w = 0$. Assume the contrary, and define

$$\hat{w} = w/|||w||| , \qquad u_\beta = (\cos\beta)v + (\sin\beta)\,\eta^{\frac{1}{2}}\hat{w} , \tag{4.9}$$

so that $|||u_\beta|||^2 = \eta$ and $|||u_\beta - v|||^2 = 2\eta(1-\cos\beta)$. Since v is a local maximizer, we may suppose that $\beta \ne 0$ and that $N_c(u_\beta) \le N_c(v)$ whenever $|\beta|$ is sufficiently small. By the definition of $\partial N_c(v)$,

$$\begin{aligned} 0 \ge N_c(u_\beta) - N_c(v) &\ge \langle g, u_\beta - v\rangle_E \\ &= \langle \alpha v + w\,,\ (\cos\beta - 1)v + (\sin\beta)\eta^{\frac{1}{2}}\hat{w}\rangle_E \\ &= \alpha(\cos\beta - 1)\eta + (\sin\beta)\eta^{\frac{1}{2}}\,|||w||| , \end{aligned}$$

and this is a contradiction for $\beta > 0$ and sufficiently small.

To prove that $\alpha \ge 0$, we let w be any element of $(\text{span}\,\{v\})^\perp\backslash\{0\}$, not related to g , and again define u_β by (4.9). Then

$$0 \ge N_c(u_\beta) - N_c(v) \ge \langle \alpha v, u_\beta - v\rangle_E = \alpha(\cos\beta - 1)\eta ,$$

which shows that $\alpha \ge 0$.

Finally, suppose that $\alpha = 0$. Then $0 \in \partial N_c(v)$, so that $N_c(v) \le N_c(u)$ for all $u \in E_c$; choosing $u = 0$, we obtain $N_c(v) = 0$. Hence $N_c(u) = 0$ for all $u \in S_c(\eta)$ sufficiently near v, and, since $N_c(0) = 0$ and N_c is

non-negative and convex, we have $N_c(u) = 0$ in the cone

$$K_\delta = \{u \in E_c \mid 0 < |||u|||^2 \le \eta,\ |||\eta^{1/2}\hat{u} - v||| < \delta\} \qquad (\hat{u} = u/|||u|||),$$

for some $\delta > 0$. This can be shown false; we set $u = \frac{1}{2}(v + w)$, choose a point $(r_0, z_0) \in \Pi$ outside the set $\{(r,z) \mid v(r,z) \le -1\}$ for some representative v of $[v] \in E_c$, and choose w as follows: $|||w|||$ is so small that $u \in K_\delta$, but $w(r,z) \to \infty$ as $(r,z) \to (r_0, z_0)$. (For example, w could be $1/r^2$ times the function in (3.4) of [12].) Then $u \in K_\delta$ but $N_c(u) > 0$. Q.e.d.

THEOREM 4.5. <u>Let</u> v <u>be a local maximizer of</u> N_c <u>on</u> $S_c(\eta)$. <u>Then</u> v <u>is a transformed weak solution of Hill's problem; in fact,</u>

$$\int_{\mathbb{R}^5} \nabla u \cdot \nabla v \, dx = \frac{1}{\alpha} \int_{P(v)} u \, dx \qquad \underline{\text{for all}} \quad u \in E, \tag{4.10}$$

<u>where</u> α <u>is as in Lemma</u> 4.4.

<u>Proof.</u> Combining the results of Lemmas 4.2, 4.3 and 4.4, we obtain

$$\int_{P(v)} u + \int_{X(v)} u_- \le \alpha \int_{\mathbb{R}^5} \nabla u \cdot \nabla v \le \int_{P(v)} u + \int_{X(v)} u_+ \tag{4.11}$$

for all $u \in E_c$. Define $P_0(v) = P(v) \cup X(v) = \{x \mid v(x) \ge \frac{1}{2}W\}$, and note that our bound (2.4) for $|P(v)|_5$ applies equally well to $P_0(v)$. Also, $u_+ \in E$ when $u \in E$, with $|||u_+||| \le |||u|||$, and similarly for u_-. Therefore, we may first use Lemma 2.3 to extend (4.11) to all $u \in E$ (just as in the proof of Theorem 2.4), and then repeat the proof of Lemma 3.1; the bounds implied by (4.11) are as adequate as were those implied by (3.1). In particular, the previous arguments show that $-\alpha\Delta v(x) = 1$ in $P(v)$, that $\Delta v(x) = 0$ wherever $v(x) < \frac{1}{2}W$, and that $P_0(v)$ is bounded; hence

$$\alpha \int_{\mathbb{R}^5} \nabla\phi \cdot \nabla v = -\alpha \int_{\mathbb{R}^5} \phi \Delta v$$

$$= \int_{P(v)} \phi - \alpha \int_{X(v)} \phi \Delta v \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}^5) .$$

But $v(x)$ is constant almost everywhere on $X(v)$; then two applications, first to ∇v and then to Δv, of a known theorem ([19], p.53) show that $\Delta v(x) = 0$ almost everywhere on $X(v)$. Thus

$$\int_{\mathbb{R}^5} \nabla\phi \cdot \nabla v = \frac{1}{\alpha} \int_{P(v)} \phi \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}^5) ,$$

and we extend this result by continuity to obtain (4.10). Q.e.d.

Remark. One can also show that if w is a local minimizer of N on $S_c(\eta)$, then $N(w) = 0$, so that $w(x) \le \frac{1}{4}W$ almost everywhere.

5. Acknowledgement

The main result of this paper was obtained while both authors enjoyed the kind hospitality of the Mathematics Research Center, University of Wisconsin, Madison, during the spring of 1983.

Appendix A. Extension to weak solutions of a theorem of Gidas, Ni and Nirenberg

The theorem in question is Theorem 2.1 of [14]. As well as giving a slight extension, we correct an oversight: that the caps $Z(\mu)$ and reflected caps $Y(\mu)$, defined in (ii) below, need not be connected, so that the maximum principle can be applied only to components of such caps. The geometrical and analytical setting is as follows.

(i) Let Ω be a bounded, connected, open set in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$; of class C^1 is sufficient for Lemmas A.1 and A.2. For Theorem A.3, however, we assume that $\partial\Omega$ is of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$ which we may take to be the same Hölder exponent for $\partial\Omega$ and for the data mentioned after (A.2). (In fact, only $\partial\Omega \cap \{x | x_1 > m - \varepsilon\}$, where m is defined below and $\varepsilon > 0$, need be of class $C^{2+\alpha}$.) The outward unit normal to $\partial\Omega$ is denoted by $\nu = (\nu_1,\ldots,\nu_n)$.

(ii) Let T_μ and x^μ be as in (3.11), except that x_n replaces x_5, but now define a <u>cap</u> by $Z(\mu) = \{x \in \Omega | x_1 > \mu\}$ and the <u>reflected cap</u> by $Y(\mu) = \{x \in \mathbb{R}^n | x^\mu \in Z(\mu)\}$. Note that our earlier definitions result from replacing Ω by $\mathbb{R}^n$, but that now $Z(\mu)$ need not be connected. Indeed, $Z(\mu)$ may have infinitely many components (maximal connected subsets) even when $\partial\Omega$ is of class C^∞. We define critical positions of the reflecting hyperplane T_μ by

$$\begin{aligned}
M &= \sup\{x_1 | x \in \Omega\} = \sup\{\mu | Z(\mu) \text{ is not empty}\}, \\
k &= \inf\{\alpha | \mu \in (\alpha, M) \Longrightarrow Y(\mu) \subset \Omega\}, \\
\ell &= \inf\{\beta | \mu \in (\beta, M) \Longrightarrow \nu_1(x) > 0 \text{ for all } x \in \partial Z(\mu) \backslash T_\mu\}, \\
m &= \max\{k, \ell\}.
\end{aligned}$$

Figure 2 illustrates this notation. The following lemma states facts needed for Theorem A.3, and shows that it is consistent with remarks in [14] to call

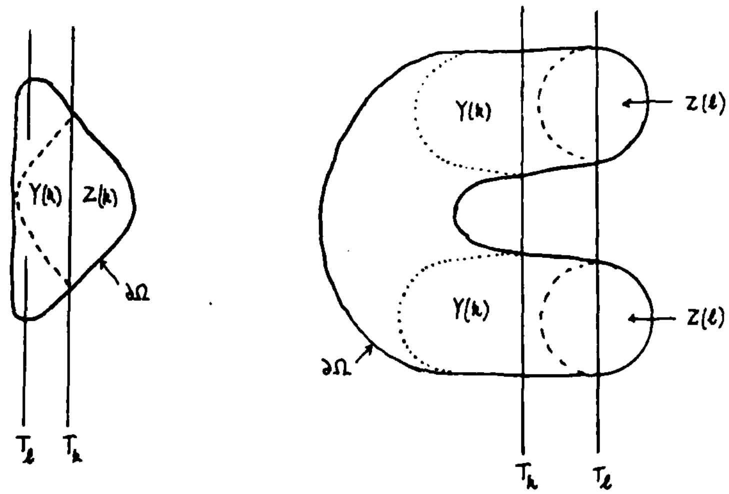


Figure 2. Some caps and reflected caps.

$Z(m)$ the maximal cap, and to call $Z(k)$, when $k < \ell$, the optimal cap. The proof of the lemma is omitted. (In [24] and [14], such results are regarded as self-evident; our proof is not difficult, but it is not short.)

LEMMA A.1. (a) *There exists a number* $\delta > 0$ *such that, for any* $\mu \in (M-\delta, M)$, *we have* $Y(\mu) \subset \Omega$ *and* $\nu_1(x) > 0$ *for all* $x \in \partial Z(\mu)\setminus T_\mu$; *hence* $k < M$ *and* $\ell < M$.

(b) $Y(k) \subset \Omega$.

(c) $\nu_1(x) > 0$ *for all* $x \in \partial Z(\ell)\setminus T_\ell$.

(d) *For any* $\mu > m$, *the boundaries* $\partial Y(\mu)\setminus T_\mu$ *and* $\partial\Omega$ *are disjoint.*

(e) *If* $k > \ell$, *then* $\partial Y(k)\setminus T_k$ *meets* $\partial\Omega$ *tangentially at some point* (*and* $\overline{Y(k)} \subset \bar{\Omega}$).

(f) *There exists a point* $x_0 \in T_\ell \cap \partial\Omega$ *such that* $\nu_1(x_0) = 0$.

Note that, if $\mu \in [m,M)$, then $Y(\mu) \subset \Omega$ and $\nu_1 > 0$ on $\partial Z(\mu)\setminus T_\mu$. The condition $k > \ell$ *in (e) is necessary.*

(iii) We consider a function $u \in C^1(\bar{\Omega})$, with $u > 0$ in Ω, such that

$$\int_\Omega \{\nabla\phi\cdot\nabla u - \phi\, b_1(x) D_1 u\}\, dx = \int_\Omega \phi g(u)\, dx \qquad \text{for all } \phi \in C_0^\infty(\Omega), \tag{A.1}$$

$$u = 0 \quad \text{on} \quad \Gamma_m, \tag{A.2}$$

where $D_1 = \partial/\partial x_1$ and $\Gamma_\mu = \partial Z(\mu)\setminus T_\mu$. We also assume that $u \in C^{2+\alpha}(\Gamma_{m-\varepsilon})$ for some $\varepsilon > 0$; that the coefficient $b_1 \in C^{0+\alpha}(\bar{\Omega})$ and that $b_1 \ge 0$ on $\overline{Y(m) \cup Z(m)}$. The function $g : [0,\infty) \to \mathbb{R}$ is assumed to have a decomposition $g = g_1 + g_2$ such that $g_1 \in C^1[0,\infty)$, while g_2 is non-decreasing and its restriction to $[0,\beta)$ is in $C^{0+\alpha}[0,\beta)$ for some $\beta > 0$.

LEMMA A.2. *Let* Ω *be as in* (i), *and* u *as in* (iii). *Assume that, for some* $\mu \in [m,M)$ *and for some component* $Y_0(\mu)$ *of* $Y(\mu)$, *we have* $u(x) \ge u(x^\mu)$

for all $x \in Y_0(\mu)$, $u(x_0) \neq u(x_0^\mu)$ *for some* $x_0 \in Y_0(\mu)$, *and* $D_1u(x) \le 0$ *for all* $x \in Z_0(\mu)$, *where* $Z_0(\mu)$ *is the reflection in* T_μ *of* $Y_0(\mu)$. *Then*

(a) $u(x) > u(x^\mu)$ *for all* $x \in Y_0(\mu)$,

(b) $D_1u(x) < 0$ *for all* $x \in \{\partial Y_0(\mu) \cap \partial Z_0(\mu)\}\backslash\partial\Omega$.

Proof. Defining $w = u_\mu - u$, where $F_\mu(x) = F(x^\mu)$ for any function F, we calculate as in the proof of Lemma 3.7. Let $Y_0 = Y_0(\mu)$ and $Z_0 = Z_0(\mu)$. Given $\phi \in C_0^\infty(Y_0)$, we first choose ϕ_μ as the test function in (A.1), transform from Z_0 to Y_0, then choose ϕ itself as the test function, to obtain (in place of (3.15))

$$\int_{Y_0} \{\nabla\phi\cdot\nabla w - \phi b_1 D_1 w\} = \int_{Y_0} \phi\{g(u_\mu) - g(u) - (b_1 + b_{1,\mu})D_1u_\mu\} \qquad \text{(A.3)}$$

for all $\phi \in C_0^\infty(Y_0)$.

Now, $g_1(u_\mu) - g_1(u) = c(x)(u_\mu - u)$ for some $c \in C(\bar\Omega)$ because $g_1 \in C^1[0,\infty)$, and $g_2(u_\mu) \le g_2(u)$ because g_2 is non-decreasing and $u_\mu \le u$ in Y_0 . By hypothesis, $b_1 + b_{1,\mu} \ge 0$ in Y_0, and $D_1u_\mu \ge 0$ in Y_0 . Hence

$$\int_{Y_0} \{\nabla\phi\cdot\nabla w - \phi b_1 D_1 w - \phi c w\} \le 0 \qquad \text{for all} \quad \phi \in C_0^\infty(Y_0) \quad \text{with} \quad \phi \ge 0 \; .$$

Since $w \not\equiv 0$ in Y_0, by hypothesis, we conclude from Theorem 3.4(a) that $w < 0$ in Y_0 . Given $x_0 \in \{\partial Y_0 \cap \partial Z_0\}\backslash\partial\Omega$, we define $B = B((x_{0,1} - R, x_0''), R)$ and choose R so small that $B \subset Y_0$. Then, since $u \in C^1(\bar\Omega)$, Theorem 3.4(b) shows that $(D_1w)(x_0) > 0$; finally, $(D_1w)(x_0) = -2(D_1u)(x_0)$. Q.e.d.

THEOREM A.3. (a) *Let* Ω *be as in* (i), *and* u *as in* (iii). *For any* $\mu \in (m,M)$, *we have* $u(x) > u(x^\mu)$ *for all* $x \in Y(\mu)$. *Also,* $D_1u(x) < 0$ *for all* x *in the maximal cap* $Z(m)$.

(b) *Suppose that* $(D_1u)(x_0) = 0$ *at some* $x_0 \in T_m \cap \Omega$. *Let* $Z_0(m)$ *be the component of* $Z(m)$ *containing* $(x_{0,1} + \varepsilon, x_0'')$ *for sufficiently small*

$\varepsilon > 0$, <u>and let</u> $Y_0(m)$ <u>be the reflection in</u> T_m <u>of</u> $Z_0(m)$. <u>Then</u> u <u>is an even function of</u> $x_1 - m$, <u>the set</u> $\bar{\Omega} = \overline{Y_0(m) \cup Z_0(m)}$, <u>and</u> $b_1 \equiv 0$ <u>in</u> Ω.

<u>Proof.</u> The hypotheses in (iii) ensure existence of a set $\Omega_{2\delta} = \{x \in \Omega | \mathrm{dist}(x,\Gamma_m) < 2\delta\}$, with $\delta > 0$, such that $g \circ u \in C^{0+\alpha}(\bar{\Omega}_{2\delta})$. Regularity theory ([2], pp. 667-8) now shows that $u \in C^{2+\alpha}(\bar{\Omega}_\delta)$. Then the proof of Lemma 2.1 in [14] stands, and Lemma A.2 replaces Lemma 2.2 of [14].

The remaining part of the proof is essentially as on pp. 218-219 of [14]. For the proof of (a), it is enough that Lemma A.2 refers to any component of $Y(\mu)$. For the proof of (b), the implication of (a) that $u(x) \geq u(x^m)$ for all $x \in Y_0(m)$, the result $D_1 u < 0$ in $Z(m)$, the hypothesis $(D_1 u)(x_0) = 0$ and Lemma A.2 imply that $u(x) = u(x^m)$ for all $x \in Y_0(m)$ (otherwise, Lemma A.2(b) would be contradicted at x_0). Then, by continuity, $u(x) = u(x^m) = 0$ for $x \in \partial Y_0(m) \backslash T_m$, and so $\partial Y_0(m) \backslash T_m \subset \partial\Omega$. Since Ω is connected, $\bar{\Omega} = \overline{Y_0(m) \cup Z_0(m)}$. To prove that $b_1 \equiv 0$ in Ω, we apply (A.3) with $\mu = m$ and $Y_0 = Y_0(m)$, noting that (A.3) follows from (A.1) without additional hypotheses. Since we now have $w = u_m - u \equiv 0$ in $Y_0(m)$, (A.3) reduces to

$$\int_{Y_0(m)} \phi(b_1 + b_{1,m}) D_1 u_m = 0 \qquad \text{for all} \quad \phi \in C_0^\infty(Y_0(m)),$$

with $D_1 u_m > 0$ by (a), and $b_1 \geq 0$, $b_{1,m} \geq 0$. Hence $b_1 \equiv 0$ and $b_{1,m} \equiv 0$ in $Y_0(m)$. Q.e.d.

<u>Appendix B. Hill's vortex in a ball</u>

The problem has been formulated in remark (ii) of section 1.5; the cross-section of the fluid domain is now $D = \{(r,z) \in \Pi | r^2 + z^2 < b^2\}$. We denote the analogue of Hill's solution by ψ_h, and let $\rho = (r^2 + z^2)^{1/2}$ as before; an elementary calculation yields

$$\psi_h(r,z) \equiv \Psi_h(r,z) + \tfrac{1}{2}Wr^2 = \begin{cases} \tfrac{1}{2}\dfrac{Wr^2}{1-c}\left(\dfrac{5}{2} - c - \dfrac{3}{2}\dfrac{\rho^2}{a^2}\right), & \rho \le a, \\ \tfrac{1}{2}\dfrac{Wr^2}{1-c}\left(\dfrac{a^3}{\rho^3} - c\right), & a \le \rho \le b, \end{cases} \tag{B.1a}$$

where

$$c = \frac{a^3}{b^3}, \qquad \frac{\lambda b^2}{W} = \frac{15}{2}\frac{1}{c^{2/3}(1-c)}, \tag{B.1b}$$

and

$$\|\psi_h\|^2 = \int_D \frac{1}{r^2}\left(\psi_{h,r}^2 + \psi_{h,z}^2\right) r\,dr\,dz = W^2 b^3 \frac{c}{1-c}\left(1 + \frac{3}{7}\frac{1}{1-c}\right). \tag{B.1c}$$

Here the norm is that of the Hilbert space $H(D)$, which results from replacing Π by D in (1.7) and in the sentence preceding it. Obviously, $\psi_h \to \psi_H$ as $b \to \infty$ with $\underline{a}$ fixed.

Suppose that W and b are prescribed. Then (B.1b) shows that $\lambda(c)$, with $0 < c < 1$, has a single stationary point, a minimum at $c = 2/5$; we define $\lambda_0 = \lambda(2/5)$. Hence, if λ is prescribed, we have no solution of Hill's type for $\lambda < \lambda_0$, one solution for $\lambda = \lambda_0$, and two solutions for $\lambda > \lambda_0$. On the other hand, the energy $\pi\|\psi_h\|^2$, as a function of c on $(0,1)$, is strictly increasing and has range $(0,\infty)$; prescription of this norm always gives exactly one solution of Hill's type.

Let b be given and fixed henceforth, and let $\Omega = B(0,b)$ in $\mathbb{R}^5$. The Hilbert spaces $E(\Omega)$ and $E_c(\Omega)$ are defined as E and E_c were, but with Ω replacing $\mathbb{R}^5$; we make the same adjustment in the definition (2.2) of $P(u)$. The transformation of weak solutions, from $H(D)$ to $E_c(\Omega)$, proceeds essentially as in section 2; perhaps a little more easily, because $E(\Omega)$ is equivalent to $\overset{\circ}{W}{}_2^1(\Omega)$ (functions in $E(\Omega)$ are in $L_p(\Omega)$ for $1 \le p \le 10/3$), and obviously $|P(u)|_5 \le |\Omega|_5$. Thus a weak solution $\psi \in H(D)$ of Hill's problem for D is equivalent, under the transformation $\psi = r^2 v$, to a function $v \in E_c(\Omega)$ satisfying the hypothesis of the following theorem. We now weaken the condition $v \in E_c(\Omega)$ to $v \in E(\Omega)$.

THEOREM B.1. _If there exist_ $v \in E(\Omega)\backslash\{0\}$, $\lambda \in \mathbb{R}$ _and_ $W > 0$ _such that_

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_{P(v)} u dx \qquad \text{for all } u \in E(\Omega), \tag{B.2}$$

then $v(x) = v_h(|x'|, x_5)$ _and_ $\lambda \ge \lambda_0$. _Here_ $v_h(r,z) = \psi_h(r,z)/r^2$ _and_ ψ_h _is as in_ (B.1); _thus_ ψ_h _denotes a pair of functions for given_ $\lambda > \lambda_0$, _but with distinct values of_ $\|\psi_h\|$.

Proof. In order to apply Theorem A.3 to v, we must first prove that $v \in C^1(\bar{\Omega})$ and that $v > 0$ in Ω. Turning to Theorem 8.1 of [1], and proceeding as in the proof of Lemma 3.1 (but now with estimates of $\|v\|_{2,p,\Omega}$), we find that $v \in C^{1+\alpha}(\bar{\Omega})$ for every $\alpha \in (0,1)$. Then $v = 0$ on $\partial\Omega$; Theorems 8.1 and 8.19 of [15] show that $v > 0$ in Ω. Alternatively, we can proceed as in the proof of Lemma 3.2 to obtain

$$v(x) = \lambda \int_{P(v)} G(x,y) \, dy \qquad \text{for all } x \in \Omega,$$

where G is the Green function of the Dirichlet problem for $-\Delta$ in the ball Ω (it is minus the function on p.19 of [15]). Classical estimates then show that $v \in C^{1+\alpha}(\bar{\Omega})$ for every $\alpha \in (0,1)$, and the positivity of v in Ω follows from that of G.

We now apply Theorem A.3 to the function v and the ball Ω in $\mathbb{R}^5$, setting $b_1 = 0$, $g_1 = 0$ and $g_2(t) = \lambda f_H(t - \frac{1}{4}W)$, where f_H is as in (1.5); the hypotheses are amply satisfied and the maximal cap is the half-ball in which $x_1 > 0$. Therefore $D_1 v(x) < 0$ whenever $x_1 > 0$ and $x \in \Omega$. Now choosing the unit vector $\gamma = (-1,0,\ldots,0)$, we see that $D_1 v(x) > 0$ whenever $x_1 < 0$ and $x \in \Omega$. By continuity, $D_1 v = 0$ on $T_0 \cap \Omega$, and the theorem now states that v is an even function of x_1. The same argument holds for every unit vector γ in $\mathbb{R}^5$; consequently, v depends only on $|x|$, and is strictly decreasing as $|x|$ increases. The proof now concludes like that of Theorem 3.9. Q.e.d.

The results in section 4 are not affected in any significant way when $\mathbb{R}^5$ is replaced by the ball Ω in $\mathbb{R}^5$, and E by $E(\Omega)$. Let us denote by $N(u,\Omega)$ and $S_c(\eta,\Omega)$ the results of these changes in the definitions (2.8) and (2.9). Then we have

THEOREM B.2. *Let* v *be a local maximizer of* $N(\cdot,\Omega)$ *on the sphere* $S_c(\eta,\Omega)$. *Then* v *is a transformed weak solution of Hill's problem for* D; *that is, there exists* $\lambda \in \mathbb{R}$ *such that* (B.2) *holds*.

Appendix C. Alternative proof of a lemma of Gidas, Ni and Nirenberg

We are concerned here with Lemma 4.1 in [14], of which our Lemma 3.5 is a particular case; the following proof is somewhat different from that in [14]. We let $x'' = (x_2,\ldots,x_n)$, as elsewhere, but r now denotes *spherical* radius.

LEMMA C.1. *Assume that, outside some ball in* $\mathbb{R}^n$,

$$v(x) = a_0 r^{-m} + g(x), \qquad a_0 > 0, \quad m > 0, \quad r \equiv |x|,$$

where $g(x) \to 0$ *and* $|\nabla g(x)| = O(r^{-m-3})$ *as* $r \to \infty$. *Consider two points* y *and* z *such that* $y_1 < z_1$, $y_1 + z_1 \geq 2\mu > 0$ *and* $y'' = z''$. *There exists a number* $R(\mu)$, *depending only on* v *and* $\min\{1,\mu\}$, *such that*

$$v(y) > v(z) \quad \textit{whenever} \quad |y| \geq R(\mu).$$

Proof. There exist positive constants r_0 and K such that, for $r \geq r_0$,

$$|\nabla g(x)| \leq a_0 K r^{-m-3}, \tag{C.1}$$

$$|g(x)| = \left|\int_r^\infty \frac{\partial g}{\partial \tilde{r}}\, d\tilde{r}\right| \leq \frac{a_0 K}{m+2} r^{-m-2}. \tag{C.2}$$

We consider only points outside the ball $B(0,r_0)$, and introduce the notation (Figure 3)

$$y_1 = \alpha - h , \qquad z_1 = \alpha + h , \qquad \alpha \geq \mu > 0 , \ h > 0 ,$$

$$s = |y| = \{(\alpha - h)^2 + |y''|^2\}^{\frac{1}{2}} , \qquad t = |z| = \{(\alpha + h)^2 + |y''|^2\}^{\frac{1}{2}} ,$$

so that

$$t^2 - s^2 = 4\alpha h , \qquad t > s \geq r_0 . \tag{C.3}$$

The result will follow from two estimates of $|g(y) - g(z)|$ and one of $s^{-m} - t^{-m}$.

Let P_2 denote the two-dimensional plane containing the points 0, y and z (or any such plane if $y'' = 0$), and let Γ be the circular arc in P_2 from y to z, centred at $(\alpha,0)$. Then Γ has length πh at most, and $r = |x| \geq s$ for $x \in \Gamma$, so that (C.1) yields

$$|g(y) - g(z)| = \left| \int_\Gamma \nabla g(x) \cdot dx \right| \leq \pi h a_0 K s^{-m-3} . \tag{C.4}$$

Alternatively, by (C.2),

$$|g(y) - g(z)| \leq |g(y)| + |g(z)| \leq \frac{2a_0 K}{m+2} s^{-m-2} . \tag{C.5}$$

For our third estimate, we first note that

$$t^m - s^m \geq c_m t^{m-1}(t-s) , \qquad \text{where} \quad c_m = \min\{m,1\} ; \tag{C.6}$$

if $m \in (0,1]$, this is true because $1 - \xi^m \geq m(1-\xi)$ for $0 \leq \xi \leq 1$ (differentiate both sides), and, if $m > 1$, because $s^m < t^{m-1}s$. Accordingly, in view of (C.3),

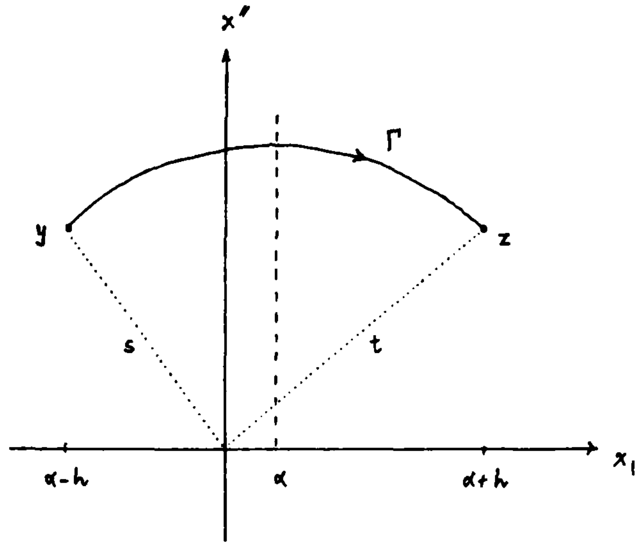


Figure 3. Notation for the proof of Lemma C.1.

$$s^{-m} - t^{-m} = \frac{t^m - s^m}{s^m t^m} \geq \frac{c_m t^{m-1}(t-s)}{s^m t^m} = \frac{c_m}{s^m t} \frac{4\alpha h}{t+s}$$

$$> \frac{2c_m \alpha h}{s^m t^2} = \frac{2c_m \alpha h}{s^m(s^2 + 4\alpha h)} . \qquad (C.7)$$

If $4\alpha h < s^2$, we use (C.7) and (C.4) to obtain

$$\frac{1}{a_0}\{v(y) - v(z)\} = s^{-m} - t^{-m} + \frac{1}{a_0}\{g(y) - g(z)\}$$

$$> \frac{c_m \alpha h}{s^{m+2}} - \frac{\pi K h}{s^{m+3}} \geq 0 \quad \text{if} \quad s \geq \frac{\pi K}{c_m \alpha} .$$

If $4\alpha h \geq s^2$, we use (C.7) and (C.5) to obtain

$$\frac{1}{a_0}\{v(y) - v(z)\} > \tfrac{1}{2}\frac{c_m}{s^m} - \frac{2K}{m+2}\frac{1}{s^{m+2}} \geq 0 \quad \text{if} \quad s^2 \geq \frac{8K}{c_m(m+2)} .$$

We define

$$R(\mu) = \max\left\{r_0 , \frac{\pi K}{c_m \min\{1,\mu\}} , \left(\frac{8K}{c_m(m+2)}\right)^{\frac{1}{2}}\right\} ,$$

and the lemma is proved. Q.e.d.

Remarks. (i) If the hypothesis $|\nabla g(x)| = O(r^{-m-3})$ as $r \to \infty$ is weakened to $|\nabla g(x)| = O(r^{-m-2-\delta})$, $\delta > 0$, then an obvious variant of the foregoing proof still holds; if it is weakened to $|\nabla g(x)| = o(r^{-m-2})$, then a proof is still possible, but explicit calculation of $R(\mu)$ must be replaced by an 'epsilon argument'.

(ii) In [14], the lemma is stated for $m > 0$ (in Theorem 4, p.211), but proved (on pp. 232-234) only for $m \geq 1$; however, (C.6) shows that this is a small matter.

References

1. AGMON, S., The L_p approach to the Dirichlet problem. Ann. Scuola Norm. Sup. Pisa, (3) 13 (1959), 405-448.

2. AGMON, S., DOUGLIS, A. and NIRENBERG, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Comm. Pure Appl. Math., 12 (1959), 623-727.

3. AMBROSETTI, A. and MANCINI, G., On some free boundary problems. In Recent contributions to nonlinear partial differential equations (edited by H. Berestycki and H. Brézis). Pitman, 1981.

4. AMICK, C.J. and FRAENKEL, L.E., The uniqueness of Norbury's perturbation of Hill's spherical vortex. To appear.

5. —— Note on the equivalence of two variational principles for steady vortex rings. To appear.

6. BENJAMIN, T.B., The alliance of practical and analytical insights into the nonlinear problems of fluid mechanics. In Applications of methods of functional analysis to problems of mechanics, Lecture notes in math. 503. Springer, 1976.

7. BERESTYCKI, H., Some free boundary problems in plasma physics and fluid mechanics. In Applications of nonlinear analysis in the physical sciences (edited by H. Amann, N. Bazley and K. Kirchgässner). Pitman, 1981.

8. CAFFARELLI, L.A. and FRIEDMAN, A., Asymptotic estimates for the plasma problem. Duke Math. J., 47 (1980), 705-742.

9. CHANDRASEKHAR, S., Hydrodynamic and hydromagnetic stability. Oxford, 1961.

10. EKELAND, I. and TEMAM, R., Convex analysis and variational problems. North-Holland, 1976.

11. ESTEBAN, M.J., Nonlinear elliptic problems in strip-like domains: symmetry of positive vortex rings. Nonlinear Analysis, Theory, Methods and Applications, 7 (1983), 365-379.

12. FRAENKEL, L.E. and BERGER, M.S., A global theory of steady vortex rings in an ideal fluid. Acta Math., 132 (1974), 14-51.

13. FRIEDMAN, A. and TURKINGTON, B., Vortex rings: existence and asymptotic estimates, Trans. Amer. Math. Soc., 268 (1981), 1-37.

14. GIDAS, B., NI, W.-M. and NIRENBERG, L., Symmetry and related properties via the maximum principle. Comm. Math. Phys., 68 (1979), 209-243.

15. GILBARG, D. and TRUDINGER, N.S., Elliptic partial differential equations of second order. Springer, 1977.

16. GILES, J.R., Convex analysis with application in differentiation of convex functions. Pitman, 1982.

17. HILL, M.J.M., On a spherical vortex. Philos. Trans. Roy. Soc. London, A 185 (1894), 213-245.

18. KEADY, G. and KLOEDEN, P.E., Maximum principles and an application to an elliptic boundary-value problem with a discontinuous nonlinearity. Research report, Dept. of Math., University of Western Australia, 1984.

19. KINDERLEHRER, D. and STAMPACCHIA, G., An introduction to variational inequalities and their applications. Academic Press, 1980.

20. NI, W.-M., On the existence of global vortex rings. J. d'Analyse Math., 37 (1980), 208-247.

21. NIRENBERG, L., On elliptic partial differential equations. Ann. Scuola Norm. Sup. Pisa, (3) 13 (1959), 115-162.

22. NORBURY, J., A steady vortex ring close to Hill's spherical vortex. Proc. Cambridge Philos. Soc., 72 (1972), 253-284.

23. —— A family of steady vortex rings. J. Fluid Mech., 57 (1973), 417-431.

24. SERRIN, J., A symmetry problem in potential theory. Arch. Rat. Mech. Anal., 43 (1971), 304-318.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER #2820	2. GOVT ACCESSION NO. AD-A158 149	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE UNIQUENESS OF HILL'S SPHERICAL VORTEX		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) C.J. Amick and L.E. Fraenkel		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - (Applied Analysis)
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE May 1985
		13. NUMBER OF PAGES 45
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

steady vortex rings, weak solutions, spherical symmetry

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The only explicit exact solution of the problem of steady vortex rings is that found, for a particular case, by M.J.M. Hill in 1894; it solves a semi-linear elliptic equation, of order two, involving a Stokes stream function $\Psi(r,z)$ and a non-linearity $f_H(\Psi)$ that has a simple discontinuity at $\Psi = 0$. In this paper we prove that (a) any weak solution of the corresponding boundary-value problem is Hill's solution, modulo translation along the axis of symmetry $(r = 0)$, (b) any solution of the isoperimetric variational problem in [12] is a

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. weak solution, indeed, any <u>local</u> maximizer is a weak solution. The result (b) is not immediate because f_H is discontinuous; consequently, the functional that is maximized is not Fréchet differentiable on the whole Hilbert space in question.

END

FILMED

10-85

DTIC